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1. Logical Operations

Fuzzy logical reasoning is a superset of standard Boolean logic. In other words, if you keep the fuzzy values at their extremes of 1 (completely true), and 0 (completely false), standard logical operations will hold. As an example, consider the following standard truth tables.

A	B	A and B
0	0	0
0	1	0
1	0	0
1	1	1

AND

A	B	A or B
0	0	0
0	1	1
1	0	1
1	1	1

OR

A	not A
0	1
1	0

NOT

Now, because in fuzzy logic the truth of any statement is a matter of degree, the input values can be real numbers between 0 and 1.

⇒ **What function might preserve the results of the AND truth table (for example) and also extend to all real numbers between 0 and 1? For example:**

A	B	A 'fuzzy and' B
0	0.3	0
0.3	0.5	0.3
0.6	0.8	0.6
1	0.8	0.8
1	1	1

⇒ **Using the same reasoning, which operation could be used to the statements A OR B and NOT A?**

Thus, you can resolve any construction using fuzzy sets and the fuzzy logical operation AND, OR, and NOT.

⇒ **Calculate** $\mu_{A\cup B}(x)$ $\mu_{A\cap B}(x)$ $\mu_{\bar{A}}(x)$ **for these two fuzzy sets:**

$$A = \left\{ \frac{0.8}{x_1} \quad \frac{0.7}{x_2} \right\} \quad B = \left\{ \frac{0.5}{x_1} \quad \frac{0.6}{x_2} \right\}$$

2. T-norms and T-conorms

In more general terms, we're defining what are known as the fuzzy intersection or conjunction (AND), fuzzy union or disjunction (OR), and fuzzy complement (NOT).
We have defined above what we'll call the classical operators for these functions: AND = min, OR = max, and NOT = additive complement.

A T-norm operator is a binary mapping T(,) satisfying the following requirements:

1. boundary: T(0, 0) = 0, T(a, 1) = T(1, a) = a
2. monotonicity: T(a, b) <= T(c, d) if a <= c and b <= d
3. commutativity: T(a, b) = T(b, a)
4. associativity: T(a, T(b, c)) = T(T(a, b), c)

⇒ **Link each above requirement with its description:**

a) This requirement imposes the correct generalization to crisp sets.
b) It indicates that the operator is indifferent to the order of the fuzzy sets to be combined.
c) This requirement allows us to take the intersection of any number of sets in any order of pairwise groupings.
d) This one implies that a decrease in the membership values in A or B cannot produce an increase in the membership value in A intersection B.

A T-conorm (or S-norm) operator is a binary mapping S(,) satisfying

- boundary: S(1, 1) = 1, S(a, 0) = S(0, a) = a
- monotonicity: S(a, b) <= S(c, d) if a <= c and b <= d
- commutativity: S(a, b) = S(b, a)
- associativity: S(a, S(b, c)) = S(S(a, b), c)

⇒ Which norm corresponds to the min (AND) fuzzy operator?
⇒ Which norm corresponds to the max (OR) fuzzy operator?

Given these functions, you can resolve any construction using fuzzy sets and the fuzzy logical operation AND, OR, and NOT.

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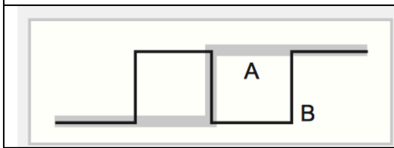
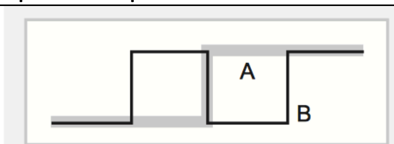
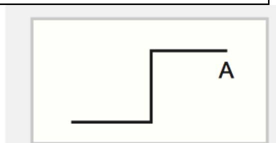
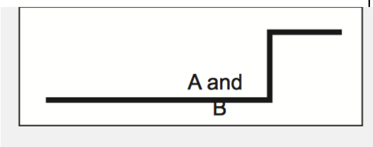
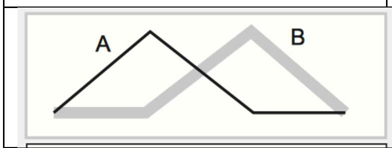
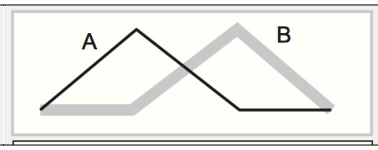
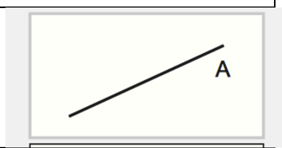
Each figure displays plots corresponding to the values of two sets, A and B.

⇒ **Try to determine in a graphical way the corresponding fuzzy operations AND, OR and NOT.**

Graphical representation		
		
A and B? 	A or B?	Not A?
		
A and B?	A or B?	Not A?

Given the following fuzzy sets:

$$A = \left\{ \frac{1}{1} + \frac{0.75}{1.5} + \frac{0.3}{2} + \frac{0.15}{2.5} + \frac{1}{3} \right\} \quad B = \left\{ \frac{1}{1} + \frac{0.6}{1.5} + \frac{0.8}{2} + \frac{1}{2.5} + \frac{0}{3} \right\}$$

Calculate:

$A \cup B \quad A \cap B \quad A \cap \overline{B} \quad \overline{A} \cap B \quad \overline{A} \cup A \quad B \cap \overline{B}$

3. If-Then Rules

Fuzzy sets and fuzzy operators are the subjects and verbs of fuzzy logic. These if-then rules are used to formulate the conditional statements that comprise fuzzy logic.
A single fuzzy if-then rule assumes the form

if *x* is *A* then *y* is *B*
antecedent => consequent (conclusion)

where *A* and *B* are linguistic values defined by fuzzy sets on the ranges (universes of discourse) X and Y, respectively. An example of such a rule might be:

If service is good then tip is average

The concept *good* is represented as a number between 0 and 1, and so the antecedent is an interpretation that returns a single number between 0 and 1. Conversely, *average* is represented as a fuzzy set, and so the consequent is an assignment that assigns the entire fuzzy set B to the output variable *y*.

Interpreting an if-then rule involves distinct parts:

1. **Evaluating the antecedent** (which involves *fuzzifying* the input and applying any necessary *fuzzy operators*)
2. **Applying that result to the consequent** (known as *implication*).

In the case of two-valued or binary logic, if-then rules do not present much difficulty. If the premise is true, then the conclusion is true. If you relax the restrictions of two-valued logic and let the antecedent be a fuzzy statement, how does this reflect on the conclusion? The answer is a simple one.

- **if the antecedent is true to some degree of membership, then the consequent is also true to that same degree.**

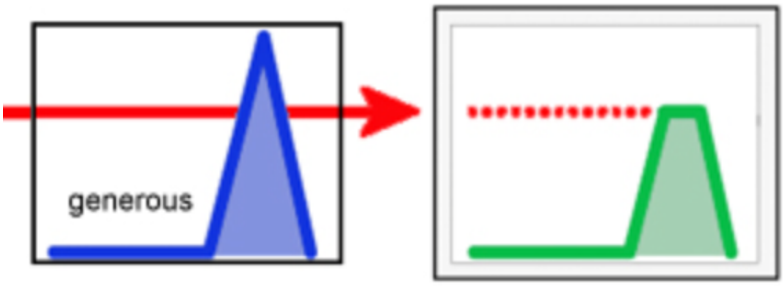
Thus:
in binary logic: $p \rightarrow q$ (p and q are either both true or both false.)
in fuzzy logic: $0.5\, p \rightarrow 0.5\, q$ (Partial antecedents provide partial implication.)

The most common way to modify the output fuzzy set is **truncation,** using the min function. Another well-known method is **scaling** using the prod function.
Let's see an example of truncation with this rule:

If the food is delicious and the service is excellent then the tip is generous

When evaluating the antecedent, suppose that the final value is 0.7, then the consequence must be truncated: $0.7\, p \rightarrow 0.7\, q$ The implication function then modifies that fuzzy set to the degree specified by the antecedent:

If the maximum degree ranged between 0 and 1, now the superior limit is decreased to 0.7.



$$\mu_A(x)=\begin{cases} 0 & x\le 20 \\ \dfrac{x-20}{25-20} & x\in(20,25] \\ \dfrac{30-x}{30-25} & x\in(25,30) \\ 0 & x\ge 30 \end{cases}$$

$$\mu_A(x)=\begin{cases} 0 & x\le 20 \\ \dfrac{x-20}{25-20} & x\in(20,23.5] \\ 0.7 & x\in(23.5,26.5] \\ \dfrac{30-x}{30-25} & x\in(26.5,30) \\ 0 & x\ge 30 \end{cases}$$

4. The Basic Tipping Problem

This example uses a two-input, one-output tipping problem based on tipping practices in the U.S. Assume that an average tip is 15%, a generous tip is 25%, and a cheap tip is 5% applied to the overall cost of the bill.

Given a number between 0 and 10 that represents the quality of service at a restaurant (where 10 is excellent), and another number between 0 and 10 that represents the quality of the food at that restaurant (again, 10 is excellent), what should the tip be?
The starting point is to write down the three golden rules of tipping:

1. ***If the service is poor or the food is rancid, then tip is cheap.***
2. ***If the service is good, then tip is average.***
3. ***If the service is excellent or the food is delicious, then tip is generous.***

STEP 1 FUZZICATION

Input Variables: Service and Food   Output Variables: Tip

Membership Functions

Find the equations corresponding to each linguistic tag

Service: { Poor, Good, Excellent}

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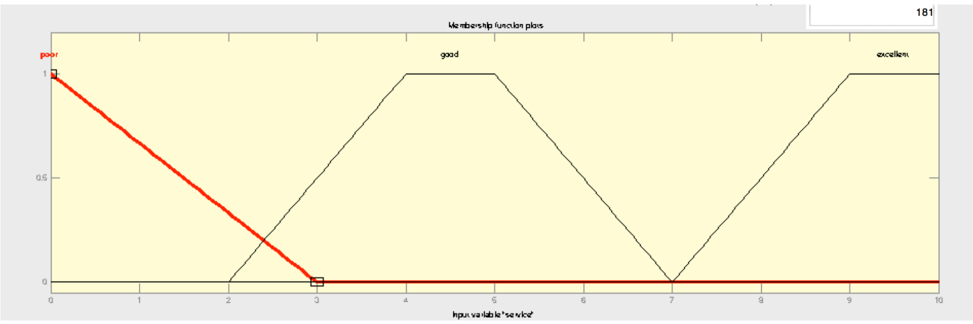
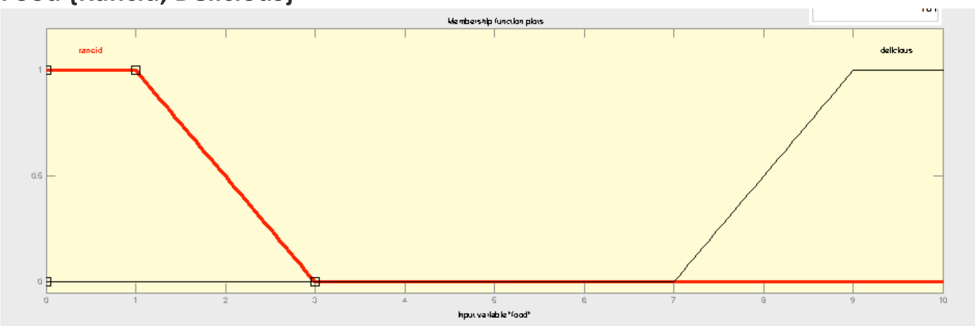


Poor=$\left\{\begin{matrix} 1 & x \leq 0 \\ \frac{3-x}{3} & x \in (0,3) \\ 0 & x \geq 3 \end{matrix}\right\}$

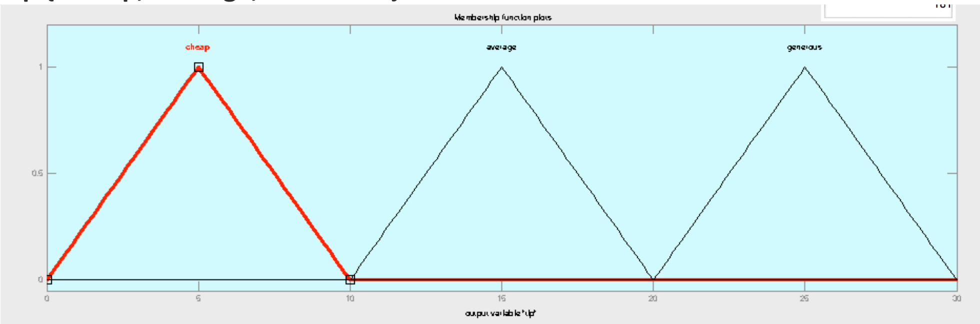
$\Rightarrow$ **Good? Excelent?**

Food {Rancid, Delicious}



$\Rightarrow$ **Rancid** **Delicious**

Tip { Cheap, Average, Generous }



⇒ **Cheap, Average, Generous**

And now consider that in a real situation, in an attempt to reconcile, Brad and Angelina went to a restaurant. They gave 8 points to the service, but just 2.5 points to the food. What tip they should have given?

STEP 2. INFERENCE

Interpreting if-then rules is a three-part process:

1 **ANTECEDENTS**

Fuzzify inputs: Resolve all fuzzy statements in the antecedent to a degree of membership between 0 and 1.

Apply fuzzy operator to multiple part antecedents: If there are multiple parts to the antecedent, apply fuzzy logic operators and resolve the antecedent to a single number between 0 and 1.

⇒ **What are the values of the antecedents of each rule after applying the crisp values service=8 and food=2.5?**

Rule1:

Rule2:

Rule3:

2 **CONSEQUENTS**: Use the degree of support for the entire rule to shape the output fuzzy set. The consequent of a fuzzy rule assigns an entire fuzzy set to the output. This fuzzy set is represented by a membership function that is chosen to indicate the qualities of the consequent. If the antecedent is only partially true, (i.e., is assigned a value less than 1), then the output fuzzy set is truncated according to the implication method.

⇒ **How is the new membership equations of the tip variable after truncating their values according to the antecedents?** (*Determine the new equations to Cheap, Average and Generous)*

3 **AGGREGATION**: Aggregation is the process of unification of the outputs of all rules.

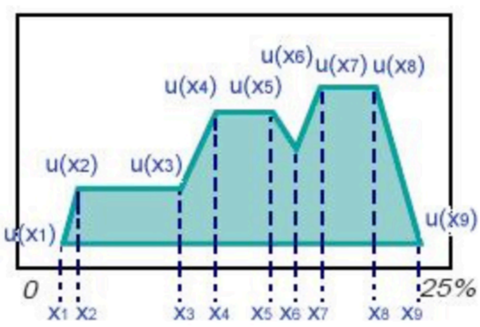
We take the membership functions of all rule consequents previously clipped or scaled and combine them into a single fuzzy set. The input of the aggregation process is the list of clipped or scaled consequent membership functions, and the output is one fuzzy set for each output variable.

⇒ **Draw the resulting aggregated output function**

STEP 3. DEFFUZIFICATION

Convert fuzzy grade to Crisp output. The last step in the fuzzy inference process is defuzzification. Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number. The input for the defuzzification process is the aggregate output fuzzy set and the output is a single number. Centroid method: finds a point representing the centre of gravity of the fuzzy set, *A*, on the interval, *a b*.

$$C = \frac{\sum_i x_i \times \mu(x_i)}{\sum_i \mu(x_i)}$$



⇒ **Calculate the centroid value**